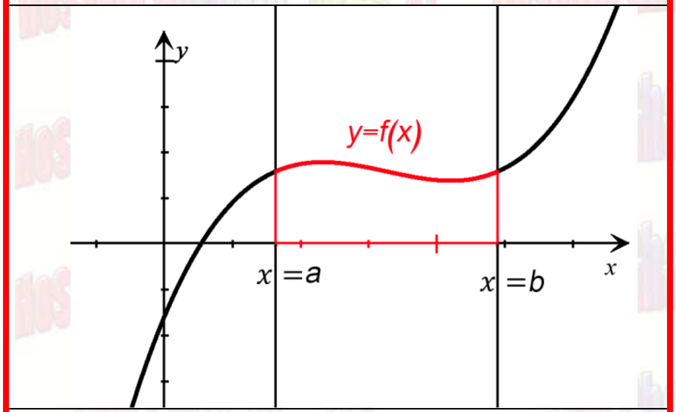
HOSSAM GHANEM

(33) 9.3 Application to Physics and Engineering
The Centroid "Center of mass"

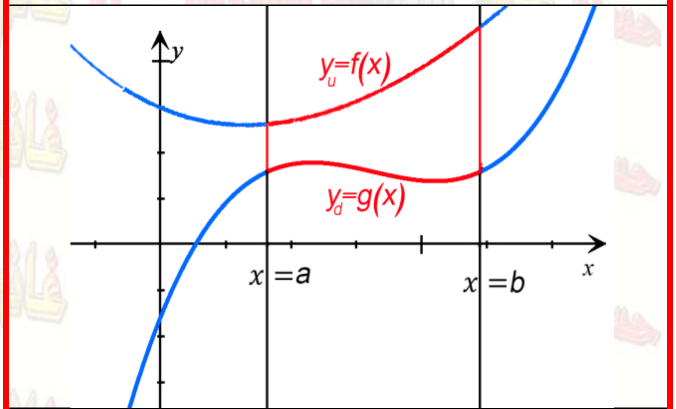
To find the Centroid of region that lies under the curve $y = f(x)$



Let the Centroid located at the point $(\overline{x}\ ,\overline{y})$

1	Find the area of the region $$A = \int_a^b f(x)\,dx$$
2	Find the $\overline{x}$ by $$\overline{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx$$
3	Find the $\overline{y}$ by $$\overline{y} = \frac{1}{2A}\int_a^b [\,f(x)]^2\,dx$$

To find the Centroid of region bounded by the curves $y = f(x)\ ,\ y = g(x)\ \ ,\ x = a$ and $x = b$



Let the Centroid located at the point $(\overline{x}\ ,\overline{y})$

1	Find the area of the region $$A = \int_a^b [f(x) - g(x)]\,dx$$
2	Find the $\overline{x}$ by $$\overline{x} = \frac{1}{A}\int_a^b x\,[f(x) - g(x)]\,dx$$
3	Find the $\overline{y}$ by $$\overline{y} = \frac{1}{2A}\int_a^b [\,f(x)]^2 - [\,g(x)]^2\,dx$$

Example 1 Find the Centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$

Solution

$$A = \int_a^b [f(x) - g(x)]\ dx = \int_0^1 (x - x^2)\ dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$\overline{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\ dx = 6\int_0^1 x(x - x^2)\ dx = 6\int_0^1 (x^2 - x^3)\ dx = 6\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$

$$= 6\left[\frac{1}{3} - \frac{1}{4}\right] = 6\cdot\left(\frac{4-3}{12}\right) = \frac{1}{6}$$

$$\overline{y} = \frac{1}{2A}\int_a^b [[f(x)]^2 - [g(x)]^2]\ dx = \frac{1}{2\cdot\left(\frac{1}{6}\right)}\int_0^1 (x^3 - x^4)\ dx = 3\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= 3\left(\frac{1}{3} - \frac{1}{5}\right) = 3\cdot\frac{5-3}{15} = \frac{2}{5}$$

$$\therefore C\ \left(\frac{1}{6}, \frac{2}{5}\right)$$

Example 2 Find the centroid of the region bounded by the curves 24 May 2005 A

$y = 1 - x^2$, $y = 0$

Solution

intersection points

$1 - x^2 = 0$ ⇨ $x^2 = 1$ ⇨ $x = \pm 1$

$$A = \int_a^b f(x)\ dx = \int_{-1}^1 (1 - x^2)\ dx = \left[x - \frac{1}{3}x^3\right]_{-1}^1 = 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right) = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$$

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\ dx = \frac{3}{4}\int_{-1}^1 x(1 - x^2)\ dx = \frac{3}{4}\int_{-1}^1 (x - x^3)\ dx = 0$$

$$\overline{y} = \frac{1}{2A}\int_{-1}^1 [f(x)]^2\ dx = \frac{3}{2\cdot 4}\int_{-1}^1 (1 - 2x^2 + x^4)\ dx = \frac{3}{4}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{3}{4}\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{3}{4}\cdot\frac{15 + 10 - 3}{15} = \frac{3}{4}\cdot\frac{8}{15} = \frac{2}{5}$$

$$\therefore C\ \left(0, \frac{2}{5}\right)$$

Example 3

27 June 2006 A

Find the centroid (center of mass) of the region bounded by the curves

$$y = \sin x \quad , \quad y = 0 \quad , \quad x = \frac{\pi}{4} \quad \text{and} \quad x = \frac{3\pi}{4}.$$

Solution

$$A = \int_a^b f(x)\ dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x\ dx = -\Big[\cos x\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = -\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\ dx = \frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x \sin x\ dx$$

$$u = x \qquad dv = \sin x\, dx$$

$$du = dx \qquad v = -\cos x$$

$$I = uv - \int v\, du$$

$$\overline{x} = \frac{-1}{\sqrt{2}}\Big[x\cos x\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos x\ dx$$

$$= \frac{1}{\sqrt{2}}\left[\frac{-3\pi}{4}\cdot\frac{-1}{\sqrt{2}} - \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)\right] + \frac{1}{\sqrt{2}}\Big[\sin x\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}\left[\frac{3\pi}{4\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right] + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] = \frac{3\pi}{8} + \frac{\pi}{8} - 0 = \frac{4\pi}{8} = \frac{\pi}{2}$$

$$\overline{y} = \frac{1}{2A}\int_a^b [f(x)]^2\ dx = \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2 x\ dx = \frac{1}{4\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 - \cos 2x)\ dx = \frac{1}{4\sqrt{2}}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} - \frac{1}{2}(-1) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right] = \frac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} + \frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}\right] = \frac{1}{4\sqrt{2}}\left[\frac{\pi}{2} + 1\right]$$

$$C\left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)\right)$$

Example 4

30 January 2008

Find the centroid of the region R , which is bounded by the curve $y = (x+1)^{2007}$, the x-axis and the lines $x = -1$ and $x = 0$

Solution

$$A = \int_a^b f(x)\ dx = \int_{-1}^{0} (x+1)^{2007}\ dx = \frac{1}{2008}\Big[(x+1)^{2008}\Big]_{-1}^{0} = \frac{1}{2008}[1-0] = \frac{1}{2008}$$

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\ dx = 2008 \int_{-1}^{0} x(x+1)^{2007}\ dx$$

$u = x \qquad dv = (x+1)^{2007}$

$du = dx \qquad v = \frac{1}{2008}(x+1)^{2008}$

$$I = uv - \int v\, du$$

$$\overline{x} = 2008\left[\frac{x}{2008}(x+1)^{2008}\right]_{-1}^{0} - 2008\cdot\frac{1}{2008}\int_{-1}^{1}(x+1)^{2008}\ dx$$

$$= 0 - \frac{1}{2009}\left[(x+1)^{2009}\right]_{-1}^{0} = -\frac{1}{2009}[1-0] = \frac{-1}{2009}$$

$$\overline{y} = \frac{1}{2A}\int_a^b [f(x)]^2\, dx = \frac{2008}{2}\int_{-1}^{0}(x+1)^{4014}\, dx = 1004\cdot\frac{1}{4015}\left[(x+1)^{4015}\right]_{-1}^{0} = \frac{1004}{4015}[1-0] = \frac{1004}{4015}$$

$$C\ \left(\frac{-1}{2009}, \frac{1004}{4015}\right)$$

Example 5

31 August 2008 A

Find the centroid of the region bounded by the curves :

$y = x^3$, $x + y = 2$, and $x = 0$

Solution

Intersection points

$x^3 = 2 - x \quad \rightarrow \quad x = 1$

$$A = \int_a^b [f(x) - g(x)]\ dx = \int_0^1 (2 - x - x^3)\ dx = \left[2x - \frac{1}{2}x^2 - \frac{1}{4}x^3\right]_0^1 = 2 - \frac{1}{2} - \frac{1}{4} = \frac{8-2-1}{4} = \frac{5}{4}$$

$$\overline{x} = \frac{1}{A}\int_a^b x\cdot[f(x) - g(x)]\ dx = \frac{4}{5}\int_0^1 x(2 - x - x^3)dx = \frac{4}{5}\int_0^1 (2x - x^2 - x^4)dx$$

$$= \frac{4}{5}\left[x^2 - \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{4}{5}\left(1 - \frac{1}{3} - \frac{1}{5}\right) = \frac{4}{5}\cdot\frac{15-5-3}{15} = \frac{4}{5}\cdot\frac{7}{15} = \frac{28}{75}$$

$$\overline{y} = \frac{1}{2A}\int_a^b [[f(x)]^2 - [g(x)]^2]\ dx = \frac{1}{2}\cdot\frac{4}{5}\int_0^1 (2-x)^2 - x^6\ dx = \frac{2}{5}\int_0^1 (4 - 4x + x^2 - x^6)\ dx$$

$$= \frac{2}{5}\left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{7}x^7\right]_0^1 = \frac{2}{5}\left[4 - 2 + \frac{1}{3} - \frac{1}{7}\right] = \frac{2}{5}\left[2 + \frac{1}{3} - \frac{1}{7}\right] = \frac{2}{5}\cdot\frac{42+7-3}{21} = \frac{2}{5}\cdot\frac{46}{21} = \frac{92}{105}$$

$$C\ \left(\frac{28}{75}, \frac{92}{105}\right)$$

Example 6

34 August 2009 A

Find the centroid of the region bounded by the curves

$$y = \sec^2 x \quad , y = 0 \, , \quad x = 0 \quad , \quad x = \pi/4$$

Solution

$$A = \int_a^b f(x)\, dx = \int_0^{\frac{\pi}{4}} \sec^2 x \; dx = \Big[\tan x\Big]_0^{\frac{\pi}{4}} = 1 - 0 = 1$$

$$\overline{x} = \frac{1}{A} \int_a^b x\; f(x)\, dx = \int_0^{\frac{\pi}{4}} x \sec^2 x \; dx$$

$u = x \qquad\qquad dv = \sec^2 x \; dx$

$du = dx \qquad\qquad v = \tan x$

$$\overline{x} = \Big[x\; \tan x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \;\; dx = \left[\frac{\pi}{4} - 0\right] + \Big[\ln(\cos x)\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}} - \ln 1 = \frac{\pi}{4} - \ln\sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$\overline{y} = \frac{1}{2A} \int_a^b [f(x)]^2 \; dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} \sec^4 x \;\; dx = \frac{1}{2} \int_0^{\pi/4} \sec^2 x \cdot \sec^2 x \;\; dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 + \tan^2 x) \cdot \sec^2 x \;\; dx$$

$t = \tan x \qquad dt = \sec^2 x$

$x = 0 \quad ⇨ \quad t = 0$

$x = \frac{\pi}{4} \quad ⇨ \quad t = 1$

$$\overline{y} = \frac{1}{2} \int_0^1 (1 + t^2)\; dt = \frac{1}{2} \left[t + \frac{1}{2}\, t^2\right]_0^1 = \frac{1}{2}\left(1 + \frac{1}{2} - 0\right) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$$

$$C \; \left(\frac{\pi}{4} - \frac{1}{2}\ln 2 \,, \frac{3}{4}\right)$$

Homework

1	Find the Centroid of the region bounded by the curves $y = 1 - x^2$, $y = 0$	
2	Find the Centroid of the region bounded by the curves $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$	
3	Find the centroid of the region bounded by the curves $y = 1/(x^2 - 2x + 5)$, $y = 0, x = -1$ and $x = 3$ (4 points)	37 August 7, 2010
4	(4 pts.) Find the coordinates of the centroid of the region bounded by the curves $x = 5 - y^2$ and $x = 0$	38 Jan. 22, 2011
5	Find the centroid of the region bounded by the graphs of $y = \text{sech}^2 x$, $y = 0$, $x = 0$ and $x = \ln\sqrt{2}$ (4 pts)	14 June 4 , 2011
6	(5 pts) Find the y −coordinate of the centroid of the region bounded by the curves $y = \frac{1}{x^2 + 1}$ and $y = \frac{1}{x + 1}$	40 August 7 , 2011